Spacetime Path Formalism: Localized States

Ed Seidewitz
seidewitz@mailaps.org

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1 Introduction

This note is an addendum to [9]. In that paper, I present a formalism for relativistic quantum mechanics in which the spacetime paths of particles are considered fundamental. These paths are parameterized by an invariant "fifth parameter" (in addition to the four coordinates of spacetime), which places the formalism within the class of "parameterized" or "worldline" approaches to quantum mechanics (see the references cited in [9]).

One of my goals in [9] was to show how the standard results of the traditional formulation of relativistic quantum mechanics and quantum field theory can be reproduced in the new formalism. This included the emergence of on-shell particle states from the naturally off-shell foundations of the formalism, along with the usual inner product between wave functions of such states.

Now, it is well known that there are issues with the ability to localize the position of particles in the usual formulation of relativistic quantum mechanics (see [5, 8]). This clearly should be addressed in any approach based on particle paths (see also, for example, [6, 7]). The intent of the present note is to show how it may be addressed in the context of formalism proposed in [9].

2 Localized States

The *path* of a particle through spacetime is an arbitrary curve parameterized by the *path parameter* λ. Note that there is no *a priori* requirement that such a curve is timelike or lightlike. Indeed, the path may cross arbitrarily forwards or backwards in time.

Let $\hat{X}^\mu(\lambda)$, for $\mu = 0, 1, 2, 3$, be position operators for a particle at the point along its path with parameter value λ. These operators have eigenstates

$$\hat{X}^\mu(\lambda)|x;\lambda\rangle = x^\mu|x;\lambda\rangle\,,$$

for each λ, with the normalization

$$\langle x';\lambda|x;\lambda\rangle = \delta^4(x'-x)\,.$$

The corresponding 4-momentum states are given by the Fourier transform

$$|p;\lambda\rangle \equiv (2\pi)^{-2} \int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}p\cdot x} |x;\lambda\rangle \,. \tag{1}$$

The position states $|x;\lambda\rangle$ are localized in 4-dimensional spacetime in the same sense that non-relativistic position states are localized in 3-dimensional space. They even evolve in λ according to a generalized Schrödinger equation (see [9], Eq. (2.21)). However, these states are essentially "virtual": they are off-shell, they do not satisfy the Klein-Gordon equation and they depend on the arbitrary path parameter λ.

Now, as shown in [9], the state of a particle (+) or antiparticle (-) with a *3-momentum* $\boldsymbol{p}$ at time t is given by

$$|t,\boldsymbol{p}_\pm\rangle = \pm(2\omega_{\boldsymbol{p}})^{-1} \int_{\mp\infty}^{t} \mathrm{d}t_0\, |t_0,\boldsymbol{p}_\pm;\lambda_0\rangle \,,$$

where $\omega_{\boldsymbol{p}} \equiv \sqrt{\boldsymbol{p}^2 + m^2}$ and

$$|t_0,\boldsymbol{p}_\pm;\lambda_0\rangle \equiv (2\pi)^{-3/2} \int \mathrm{d}^3x\, \mathrm{e}^{\mathrm{i}(\mp\omega_{\boldsymbol{p}} t_0 + \boldsymbol{p}\cdot\boldsymbol{x})} |t_0,\boldsymbol{x};\lambda_0\rangle \,, \tag{2}$$

with λ_0 being an arbitrary reference value for the path parameter(see [9], Eqs. (2.35) and (2.36)). On-shell particle and antiparticle states are then given by

$$|\boldsymbol{p}_\pm\rangle \equiv \lim_{t\to\pm\infty} |t,\boldsymbol{p}_\pm\rangle = (2\pi)^{1/2}(2\omega_{\boldsymbol{p}})^{-1} |\pm\omega_{\boldsymbol{p}},\boldsymbol{p};\lambda_0\rangle \,. \tag{3}$$

Note that these states are similar to the "mass representation" states of [6].

Unfortunately, the states defined in Eq. (3) are not normalizable using the usual inner product, since

$$\langle \boldsymbol{p}'_\pm | \boldsymbol{p}_\pm \rangle = 2\pi(2\omega_{\boldsymbol{p}})^{-2}\delta(0)\delta^3(\boldsymbol{p}' - \boldsymbol{p})$$

is infinite. In [6], this is handled by allowing the mass m to vary, even though the energy is fixed at $\sqrt{\boldsymbol{p}^2 + m^2}$. In [9], I take a different approach, noting that

$$\langle \boldsymbol{p}_\pm | t_0, \boldsymbol{p}_{0\pm};\lambda_0\rangle = (2\omega_{\boldsymbol{p}})^{-1}\delta^3(\boldsymbol{p} - \boldsymbol{p}_0) \,, \tag{4}$$

for *any* value of t_0 (see [9], Eq. (2.39)). This essentially provides the basis for an "induced" inner product, in the sense of [3, 4].

To see how this induced inner product may be used, consider, the two Hilbert-space subspaces spanned by the normal particle states $|t,\boldsymbol{p}_+;\lambda_0\rangle$ and the antiparticle states $|t,\boldsymbol{p}_-;\lambda_0\rangle$, for each time t. States in these subspaces have the form

$$|t,\psi_\pm\rangle \equiv \int \mathrm{d}^3p\, \psi(\boldsymbol{p}) |t,\boldsymbol{p}_\pm;\lambda_0\rangle \,, \tag{5}$$

for any square-integrable function $\psi(\boldsymbol{p})$ (the notation here is slightly different than the notation $|\psi_\pm;\lambda_0\rangle$ used in [9], in order to emphasize the time dependence). Similarly, consider the dual subspaces spanned by the bra states $\langle\boldsymbol{p}_+|$ and $\langle\boldsymbol{p}_-|$, such that

$$\langle\psi_\pm| \equiv \int \mathrm{d}^3p\, \psi(\boldsymbol{p})^* \langle\boldsymbol{p}_\pm|\,, \tag{6}$$

As a result of Eq. (4), we get the traditional inner product

$$\langle\psi_\pm|t,\psi_\pm\rangle = \int \frac{\mathrm{d}^3p}{2\omega_{\boldsymbol{p}}} \psi(\boldsymbol{p})^*\psi(\boldsymbol{p})\,. \tag{7}$$

With the inner product given by Eq. (7), the spaces of the $|t,\psi_\pm\rangle$ can be seen as "reduced" Hilbert spaces in their own right, with the dual Hilbert space being the spaces of the $\langle\psi_\pm|$. With respect to these Hilbert spaces, the matrix element for the 3-position operator $\hat{\boldsymbol{X}}(\lambda_0)$ is

$$\begin{aligned}\langle\psi'_\pm|\hat{\boldsymbol{X}}(\lambda_0)|t,\psi_\pm\rangle &= \int \mathrm{d}^3p' \int \mathrm{d}^3p\, \psi'(\boldsymbol{p}')^*\psi(\boldsymbol{p})\langle\boldsymbol{p}'_\pm|\hat{\boldsymbol{X}}(\lambda_0)|t,\boldsymbol{p}_\pm;\lambda_0\rangle \\ &= (2\pi)^{-3/2} \int \mathrm{d}^3p' \int \mathrm{d}^3p\, \psi'(\boldsymbol{p}')^*\psi(\boldsymbol{p}) \int \mathrm{d}^3x\, \mathrm{e}^{\mathrm{i}(\mp\omega_{\boldsymbol{p}}t+\boldsymbol{p}\cdot\boldsymbol{x})}\boldsymbol{x}\langle\boldsymbol{p}'_\pm|t,\boldsymbol{x};\lambda_0\rangle\,.\end{aligned}$$

Now, using Eqs. (1) and (3),

$$\begin{aligned}\langle\boldsymbol{p}_\pm|t,\boldsymbol{x};\lambda_0\rangle &= (2\pi)^{1/2}(2\omega_{\boldsymbol{p}})^{-1}\langle\pm\omega_{\boldsymbol{p}},\boldsymbol{p};\lambda_0|t,\boldsymbol{x};\lambda_0\rangle \\ &= (2\pi)^{-3/2}(2\omega_{\boldsymbol{p}})^{-1}\mathrm{e}^{\mathrm{i}(\pm\omega_{\boldsymbol{p}}t-\boldsymbol{p}\cdot\boldsymbol{x})}\,.\end{aligned}$$

Then,

$$\begin{aligned}\langle\psi'_\pm|\hat{\boldsymbol{X}}(\lambda_0)|t,\psi_\pm\rangle &= (2\pi)^{-3} \int \mathrm{d}^3p' \int \mathrm{d}^3p\, \psi'(\boldsymbol{p}')^*\psi(\boldsymbol{p})(2\omega_{\boldsymbol{p}'})^{-1} \int \mathrm{d}^3x\, \mathrm{e}^{\mathrm{i}[\mp(\omega_{\boldsymbol{p}}-\omega_{\boldsymbol{p}'})t+(\boldsymbol{p}-\boldsymbol{p}')\cdot\boldsymbol{x}]}\boldsymbol{x} \\ &= (2\pi)^{-3} \int \frac{\mathrm{d}^3p'}{2\omega_{\boldsymbol{p}'}} \psi'(\boldsymbol{p}')^*\mathrm{e}^{\pm\mathrm{i}\omega_{\boldsymbol{p}'}t}\,\mathrm{i}\frac{\partial}{\partial\boldsymbol{p}'}\left[\int \mathrm{d}^3p\, \psi(\boldsymbol{p})\mathrm{e}^{\mp\mathrm{i}\omega_{\boldsymbol{p}}t} \int \mathrm{d}^3x\, \mathrm{e}^{\mathrm{i}(\boldsymbol{p}-\boldsymbol{p}')\cdot\boldsymbol{x}}\right] \\ &= \int \frac{\mathrm{d}^3p'}{2\omega_{\boldsymbol{p}'}} \psi'(\boldsymbol{p}')^*\mathrm{e}^{\mp\mathrm{i}\omega_{\boldsymbol{p}'}t}\,\mathrm{i}\frac{\partial}{\partial\boldsymbol{p}'}\left[\mathrm{e}^{\pm\mathrm{i}\omega_{\boldsymbol{p}'}t}\psi(\boldsymbol{p}')\right]\,.\end{aligned}$$

Thus, in the momentum representation, the 3-position operator is given by

$$\mathrm{e}^{\mp\mathrm{i}\omega_{\boldsymbol{p}}t}\,\mathrm{i}\frac{\partial}{\partial\boldsymbol{p}}\mathrm{e}^{\pm\mathrm{i}\omega_{\boldsymbol{p}}t}\,. \tag{8}$$

But this is just the traditional momentum representation for the 3-position operator, $\mathrm{i}\partial/\partial\boldsymbol{p}$, translated to time t.

3 Discussion

In contrast to the results of the previous section, Newton and Wigner [8] conclude that a localized particle wave function satisfying the Klein-Gordon equation is an eigenfunction of

$$\mathrm{i}\left(\frac{\partial}{\partial \boldsymbol{p}} - \frac{\boldsymbol{p}}{2\omega_{\boldsymbol{p}}^2}\right),$$

which has an extra term over the expected $\mathrm{i}\partial/\partial\boldsymbol{p}$. The key reason for this difference is the construction of the states in Eqs. (5) and (6). We can rewrite Eq. (4) as an orthonormality relation

$$(2\omega_{\boldsymbol{p}})\langle \boldsymbol{p}_\pm | t_0, \boldsymbol{p}_{0\pm}; \lambda_0\rangle = \delta^3(\boldsymbol{p} - \boldsymbol{p}_0)\,. \tag{9}$$

A wave function $\psi(\boldsymbol{p})$ in the reduced Hilbert spaces is then given by

$$\psi(\boldsymbol{p}) = (2\omega_{\boldsymbol{p}})\langle \boldsymbol{p}_\pm | \psi\rangle\,.$$

Now, inverting Eq. (2) gives

$$|t, \boldsymbol{x}; \lambda_0\rangle \equiv (2\pi)^{-3/2} \int \mathrm{d}^3x\, \mathrm{e}^{\mathrm{i}(\pm\omega_{\boldsymbol{p}} t - \boldsymbol{p}\cdot\boldsymbol{x})} |t, \boldsymbol{p}_\pm; \lambda_0\rangle\,,$$

which shows that the position eigenstate $|t, \boldsymbol{x}; \lambda_0\rangle$ is in the reduced Hilbert spaces. Its wave function is then, using Eq. (9),

$$(2\omega_{\boldsymbol{p}})\langle \boldsymbol{p}_\pm | t, \boldsymbol{x}; \lambda_0\rangle = (2\pi)^{-3/2} \mathrm{e}^{\mathrm{i}(\pm\omega_{\boldsymbol{p}} t - \boldsymbol{p}\cdot\boldsymbol{x})}\,.$$

This is just a plane wave, and it is an eigenfunction of the operator (8) with eigenvalue $\boldsymbol{x}$.

In contrast, the traditional formalism does not make a clear distinction between the on-shell ket states $|\psi_\pm\rangle$ and the time-dependent states $|t, \psi_\pm; \lambda_0\rangle$. Therefore, a symmetric normalization is usually used for the 3-momentum basis states, corresponding to $(2\omega_{\boldsymbol{p}})^{1/2}|t, \boldsymbol{p}_\pm; \lambda_0\rangle$ and $(2\omega_{\boldsymbol{p}})^{1/2}\langle \boldsymbol{p}_\pm|$ in the present formalism. This transformation from the basis used in Eq. (9) preserves the inner product

$$(2\omega_{\boldsymbol{p}})^{1/2}\langle \boldsymbol{p}_\pm | t_0, \boldsymbol{p}_{0\pm}; \lambda_0\rangle (2\omega_{\boldsymbol{p}})^{1/2} = \delta^3(\boldsymbol{p} - \boldsymbol{p}_0)\,.$$

Using this basis, the wave function of $|t, \boldsymbol{x}; \lambda_0\rangle$ is

$$(2\omega_{\boldsymbol{p}})^{1/2}\langle \boldsymbol{p}_\pm | t, \boldsymbol{x}; \lambda_0\rangle = (2\pi)^{-3/2}(2\omega_{\boldsymbol{p}})^{-1/2} \mathrm{e}^{\mathrm{i}(\pm\omega_{\boldsymbol{p}} t - \boldsymbol{p}\cdot\boldsymbol{x})}\,. \tag{10}$$

At $t = 0$ this is exactly the Newton-Wigner wave function for a localized particle [8].

Note that the basis in Eq. (9) can be obtained from that in Eq. (10) by a scalar Foldy-Wouthuysen transformation [1, 2][1]. This makes sense, since the Foldy-Wouthuysen transformation produces a representation that separates positive and negative energy states (particles and antiparticles) and gives a reasonable non-relativistic limit.

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[1] I am indebted to Prof. Larry Horwitz of Tel Aviv University for pointing this out to me.

Indeed, from Eq. (2) we can easily see that the time evolution of the 3-momentum states $|t, \boldsymbol{p}_{\pm}; \lambda_0\rangle$ is given by

$$\mathrm{e}^{\mathrm{i}\hat{P}^0 \Delta t}|t, \boldsymbol{p}_{\pm}; \lambda_0\rangle = \mathrm{e}^{\pm \mathrm{i}\omega_{\boldsymbol{p}} \Delta t}|t + \Delta t, \boldsymbol{p}_{\pm}; \lambda_0\rangle = \mathrm{e}^{\pm \mathrm{i}\hat{H}_{FW} \Delta t}|t + \Delta t, \boldsymbol{p}_{\pm}; \lambda_0\rangle \,,$$

where

$$\hat{H}_{FW} = (\hat{\boldsymbol{P}} \cdot \hat{\boldsymbol{P}} + m^2)^{1/2}$$

is the scalar Foldy-Wouthuysen Hamiltonian and the $\hat{P}^{\mu}$ are the generators of spacetime translations. Taking

$$|t + \Delta t, \psi_{\pm}\rangle = \mathrm{e}^{\mathrm{i}\hat{P}^0 \Delta t}|t, \psi_{\pm}\rangle \,,$$

Eq. (5) then implies that time-dependent wave functions should evolve as

$$\psi(t + \Delta t, \boldsymbol{p}) = \mathrm{e}^{\pm \mathrm{i}\omega_{\boldsymbol{p}} \Delta t}\psi(t, \boldsymbol{p}) \,.$$

In the non-relativistic limit, for positive-energy particles, this reduces to time evolution according to the usual non-relativistic Hamiltonian (up to an unimportant phase dependent on the particle's rest mass).

References

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